Lecture 2. ***Concept of controllability. Controllability criteria R. Kallman and E. Gilbert***

2.1. Controllability in the linearized Control Systems

In analysis and synthesis of linear control systems an important role belongs to the term of controllability. OC may have or may not have it. It is clear that if CO has a property of controllability, then the law of control may be synthesized for it. Without taking into account the properties of controllability an error may be committed in defining order of object, in defining stability of the system, especially if for analysis and synthesis a device of transfer functions is used in terms "input-output". Usage of object description in state space lets to avoid these errors and, besides, to formulate criterion, by which the controllability of object may be characterized beforehand.

Let us explain it by means of the following example.

Example 5.5. Suppose the system of following type is specified (Fig. 5.32).

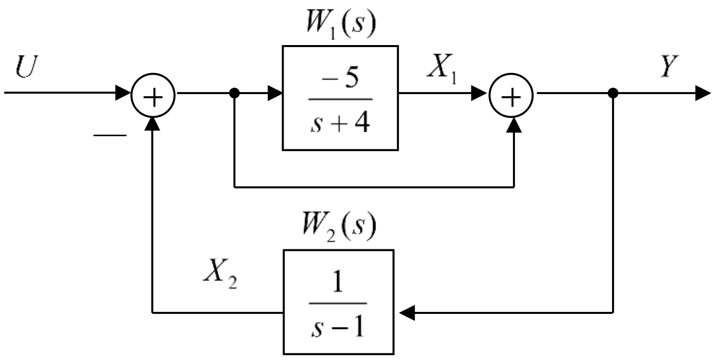

Fig. 5.32. Structural scheme of investigated system

Direct calculations give the following transfer function of a closed-loop system:

$$W_{CLOSED}(s) \overset{\Delta}{=} \frac{Y(s)}{U(s)} = \frac{\left(\frac{-5}{s+4}+1\right)}{1+\left(\frac{1}{s-1}\right)\left(\frac{-5}{s+4}+1\right)} = \frac{(-5+s+4)}{(s+5)} = \frac{s-1}{s+5}.$$

Analysis of obtained transfer function of the system shows that system order is equal to one, i.e. n = 1. Transfer function of the system has a pole $s=-5$, laid in left part of complex plane. Consequently, we may come to conclusion that the system is *asymptotically stable*.

But such conclusion is not right. We may make sure in it by making description of the system in space of states $x_1,\ x_2$.

System of differential equations of the first order in the form of Laplas's transformation for this structure (Fig. 5.32) is given by:

$$\begin{cases} x_1(s) = \dfrac{-5}{s+4}(u(s) - x_2(s)) \\ x_2(s) = \dfrac{1}{s-1}(x_1(s) + (u(s) - x_2(s))) \\ y(s) = x_1(s) + (u(s) - x_2(s)) \end{cases}; \quad \begin{cases} x_1(s)(s+4) = -5(u(s) - x_2(s)) \\ x_2(s)(s-1) = (x_1(s) + (u(s) - x_2(s))) \\ y(s) = x_1(s) + (u(s) - x_2(s)) \end{cases}.$$

Using Laplas's backward transformation to obtained equations, let us write the system of differential equations:

$$\begin{cases} \dot{x}_1 = -4x_1 + 5x_2 - 5u \\ \dot{x}_2 = x_1 + u \\ y = x_1 - x_2 + u \end{cases}.$$

In matrix form system description will be written in state space in the following way:

$$\begin{vmatrix} \dot{x}_1 \\ \dot{x}_2 \end{vmatrix} = \begin{vmatrix} -4 & 5 \\ 1 & 0 \end{vmatrix} \cdot \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} + \begin{vmatrix} -5 \\ 1 \end{vmatrix} \cdot u .$$

Also let us write equation in matrix form for output variable:

$$y = (1, -1) \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} + u .$$

Generally in matrix form the system of equations in state space will have the following form:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases},$$

where $A = \begin{vmatrix} -4 & 5 \\ 1 & 0 \end{vmatrix}$, $B = \begin{vmatrix} -5 \\ 1 \end{vmatrix}$, $C = \begin{vmatrix} 1 & -1 \end{vmatrix}$, $D = 1$.

In order to make analysis of the properties of the received system, we should characterize the proper values of matrix *A*.

Characteristic equation for the system will be written in the following way:

$$\det(A-\lambda I)=0.$$

Having accomplished the equation of the system, we'll get the following:

$$\det\begin{vmatrix}(-4-\lambda) & 5\\ 1 & -\lambda\end{vmatrix}=(4+\lambda)\lambda-5=\lambda^2+4\lambda-5=0$$

$$\lambda_{1,2}=-2\pm\sqrt{4+5}$$

$$\lambda_1=-5;\quad \lambda_2=1.$$

Matrix A has its proper values $\lambda_1=-5;\ \lambda_2=1$.

Analysis of the obtained results shows that actually the investigated system:

a) has an order equal to two (by transfer function 1);

b) is instable, because $\lambda_2 > 0$ (it is defined by transfer function that the system is asymptotically stable).

The system properties were not changed. The obtained results do not contradict to each other, which will be shown below.

In order to avoid any errors in ACS synthesis, let's introduce a concept of controllability of the system and formulate a criterion of controllability.

The first concept of controllability was introduced by an American scientist Richard Kalman (1960).

Definition of controllability. Suppose the system is described with equations:

$$\begin{cases}\dot{X}=AX+BU & (5.4)\\ Y=C^T X & (5.5)\end{cases},$$

where $X(n*1)$ is a state vector; $U(m*1)$ is a control vector; $Y(r\times 1)$ is an output variables vector or measure vector; $A(n\times n)$ is a matrix, characterizing CO, $B(n\times m)$ is a matrix, characterizing control channel, $C(r\times n)$ is a matrix of observations.

A system is called *completely controlled*, if for the system there is a finite input signal $u(t)$, transferring it from any initial state X_0 to any prescribed beforehand final state X_k for termination time.

2.2 Controllability criteria

Let's view a square matrix $V(n\times n)$, formed by proper vectors of the original matrix A:

$$V=(V_1,\ V_2,\ \ldots,\ V_n);\quad (A-\lambda_i I)V_i=0\quad \forall i=\overline{1,n}$$

and called *modal* for matrix A.

Through linear transformation of coordinates $X = V \cdot X^*$, where V is a modal matrix for the original matrix A, we will transform the equations of system state and observations (5.4), (5.5) to a diagonal form:

$X = VX^*$

$V\dot{X}^* = AVX^* + BU$

$V^{-1}V\dot{X}^* = V^{-1}AVX^* + V^{-1}BU$

$$\begin{cases} \dot{X}^* = \Lambda X^* + B^* U & (5.6) \\ Y^* = C^* X^* & (5.7) \end{cases}.$$

Here $\Lambda = V^{-1}AV$, $B^* = V^{-1}B$, $C^* = CV$.

Matrix Λ is diagonal and contains proper values of matrix A at leading diagonal. Using diagonal form of the equation of the system, we may give the following criterion of controllability.

Criterion1. E. Gilbert's criterion of controllability

The criterion formulation: if matrix B^* does not contain zero rows, then the system (5.6), (5.7) is completely controlled, otherwise – it is non-controlled.

Actually, if matrix B^* contains zero rows, then control $U(t)$ doesn't impact on those components of vector X^*, which are placed on these rows. Physically this causes impossibility to get into any point of space of states through the bound control, i.e. the system is non- controllable.

Example 5.6. Suppose mathematical description of the system is prescribed in the form of equations (5.6).

$$\begin{vmatrix} \dot{x}_1^* \\ \dot{x}_1^* \end{vmatrix} = \begin{vmatrix} -1 & 0 \\ 0 & -2 \end{vmatrix} \cdot \begin{vmatrix} x_1^* \\ x_1^* \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 2 & 1 \end{vmatrix} \cdot \begin{vmatrix} u_1 \\ u_2 \end{vmatrix}.$$

It is required to characterize controllability of the system investigated.

Solving

By Gilbert's criterion of controllability: the system is non-controllable, as matrix B^* contains a zero row. Let us write the investigated system in scalar form:

$$\begin{cases} \dot{x}_1^* = -x_1^* \\ \dot{x}_2^* = -2x_2^* + 2u_1 + u_2 \end{cases}.$$

The system of equations implies that controls u_1 and u_2 are actually don't force on the variable $\dot{x}_1^*$. Consequently, the system is non-controllable.

Let us view other criterion of controllability, which doesn't require transforming the system of equations to a diagonal form.

Criterion 2. R. Kalman's criterion of controllability

Suppose the system is described with an equation:

$$\dot{x} = Ax + bu\,,$$

where u is a scalar control, i.e. $u \in R^1$.

At initial instant of time $t = 0$ the system state is specified as $x(0)$. The desired termination system state at final instant of time $t = t_k$ $(t_k > 0)$ we will indicate as $x(k)$.

Controllability determines the conditions, at which control transfers the system from initial state $x(0)$ to termination one $x(k)$. It means that for $t = 0, 1, 2, ..., k$ the following ratios are fair:

$$x(k+1) = Ax(k) + bu(k),$$

i.e.
$$\begin{aligned}
&x(1) = Ax(0) + bu(0) \\
&x(2) = Ax(1) + bu(1) = A(Ax(0) + bu(0)) + bu(1) = A^2x(0) + Abu(0) + bu(1) \\
&x(3) = Ax(2) + bu(2) = A\left(A^2x(0) + Abu(0) + bu(1)\right) + bu(2) = A^3x(0) + A^2bu(0) + Abu(1) + bu(2). \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&x(k) = A^kx(0) + A^{k-1}bu(0) + A^{k-2}bu(1) + ... + A^0bu(k-1).
\end{aligned}$$

Sure that $A^0 = A^{k-k}$.

The system of equations may be written in the following way:

$$x(k) - A^kx(0) = \left(A^{k-1}b,\ A^{k-2}b,\ ...,\ b\right)\cdot\begin{vmatrix} u(0) \\ u(1) \\ ... \\ u(k-1) \end{vmatrix}. \qquad (**)$$

For the system of equations (**) to have a solution towards variables $u(0), u(1), ..., u(k-1)$, matrix $\left(A^{k-1}b, A^{k-2}b, ..., b\right)$ should have a rank, equal to k. It follows that other criterion of controllability.

Criterion 2. R. Kalman's criterion

For complete controllability of a dynamic system (5.4) it is necessary and enough in order for rank of block matrix of controllability

$$K_u = \left(b,\ Ab,\ A^2b,\ \ldots,\ A^{n-1}b\right)$$

to be equal to system order *n*, i.e. $rank\, K_u = n$.

Formulated criterion is easy to spread on the systems with vector control. Suppose $u(l \times l)$ is a control vector. Then the matrix of controllability K_u of size $(n \times \ln)$ is given by:

$$K_u = \left(B, AB, A^2B, \ldots, A^{n-1}B\right),$$

where $B(n \times l)$ is a matrix. If from the matrix columns of controllability K_u to make all possible matrixes of size $(n \times n)$ and among them there will be at least one, the rank of which will be equal to system order *"n"*, then the system with vector control will be completely controllable, otherwise it will be non-controllable.

Example 5.7. Suppose ACS mathematical description is specified as a system of equations (5.4), where matrixes *A* and *B* are equal to:

$$A = \begin{vmatrix} 6 & 5 \\ 1 & 2 \end{vmatrix},\ B = \begin{vmatrix} -3 \\ 4 \end{vmatrix}.$$

It is required to characterize controllability of investigated system.

Algorithm and solving

We should make a block matrix of controllability and define its rank.

1) Matrix *A*(2*2), consequently, the system order is equal to two, i.e. *n* = 2. The matrix of controllability for this system will be written in the following way:

$$K_u = (B, AB).$$

2) The matrix rank of controllability is defined:

$AB = \begin{vmatrix} 6 & 5 \\ 1 & 2 \end{vmatrix}\begin{vmatrix} -3 \\ 4 \end{vmatrix} = \begin{vmatrix} 2 \\ 5 \end{vmatrix}$; $K_u = \begin{vmatrix} -3 & 2 \\ 4 & 5 \end{vmatrix}$; $\Delta_1 = -3 \neq 0$; $\Delta_2 = -23 \neq 0$;

$rank\, K_u = 2$; *n* =2. *Hence,* $rank\, K_u = n$.

The system investigated is *completely controllable*, as the rank of block matrix of controllability is equal to the system order.

According to introduced definition of controllability we may conclude the following:

Firstly, if the system is completely controllable, its description in terms “input-output” through transfer functions and equations of states are equivalent.

Secondly, if the system is non-controllable, its description through equations of states is more complete, as it takes into account controllable and non-controllable parts of the system, and description of the system through transfer functions in terms “input-output” takes into account only its controllable part.

This is exact why in the given example 5.5 there are no any contradictions.

Conclusion. Mathematical apparatuses for ACS dynamics analysis in state space and in terms “input-output” as a transfer functions supplement each other, deeply representing dynamic properties of automated control systems.